PLANE AND AXISYMMETRIC PROBLEMS OF THE ACTION OF LOADS ON A THIN NONUNIFORM VISCOELASTIC LAYER

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Problems of the action of loads on thin* nonuniform viscoelastic layers are investigated; two kinds of nonuniformity are taken into account. The first nonuniformity is characterized by the fact that the elements of the layers have different elastic and rheological properties, and the second is caused by nonuniform aging of the material. Approximate solutions of the problems are found. Different particular cases are discussed.

1. Let us consider problems of the action of loads on thin layers described by the following equations of state [1-3]:

$$\begin{gathered}
\varepsilon_{ij}(\mathbf{x}, t) = (1+\nu)(I-L)\frac{\sigma_{ij}(\mathbf{x}, t)}{E} - \delta_{ij}\nu(I-L)\frac{\sigma_{kk}(\mathbf{x}, t)}{E}, \\
(I-L)\frac{\omega(\mathbf{x}, t)}{E} = \frac{\omega(\mathbf{x}, t)}{E(t+\varkappa(\mathbf{x}), \mathbf{x})} - \int_{\tau_0}^{t}\frac{\omega(\mathbf{x}, \tau)}{E(\tau+\varkappa(\mathbf{x}), \mathbf{x})}K(t+\varkappa(\mathbf{x}), \tau+\varkappa(\mathbf{x}), \mathbf{x})\,d\tau, \\
K(t, \tau, \mathbf{x}) = E(t, \mathbf{x})\frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau, \mathbf{x})} + C(t, \tau, \mathbf{x})\right], \\
\sigma_{ij}(\mathbf{x}, t) = \frac{E(t, \mathbf{x})}{1+\nu}\left[(I+N)\varepsilon_{ij}(\mathbf{x}, t) + \delta_{ij}\frac{\nu}{1-2\nu}(I+N)\varepsilon_{kk}(\mathbf{x}, t)\right], \\
(I+N)\omega(\mathbf{x}, t) = \omega(\mathbf{x}, t) + \int_{\tau_0}^{t}\omega(\mathbf{x}, \tau)R(t+\varkappa(\mathbf{x}), \tau+\varkappa(\mathbf{x}), \mathbf{x})\,D\tau,
\end{gathered} \tag{1.1}$$

where ε_{ij} and σ_{ij} are components of the strain and stress tensors, ε_{kk} is the volume strain, $\sigma_{kk}/3$ is the average hydrostatic pressure, $E(t, x)$ is the modulus of the instantaneous elastic strain, $x(x_1, x_2, x_3)$ is the observed point of the body, t is the current time, τ_0 is the age of the body at the point with coordinates $x(0, 0, 0)$ at the time the stresses are applied, ν = const is the Poisson coefficient, $K(t, \tau, x)$ is the creep kernel, $R(t, \tau, x)$ is its resolvent, $C(t, \tau, x)$ is the measure of the creep upon tension or compression, $\varkappa(x)$ is the nonuniform aging function, and δ_{ij} is the Kronecker delta.

Next we shall investigate the case of plane strain.

Problem 1. The action of a normal load $q(x_1, t)$ on a nonuniform viscoelastic thin layer lying without friction on a rigid base.

It is necessary to add to the expressions (1.1) equilibrium equations, relations which relate the strains to the displacements, and the boundary conditions:

$$\sigma_{11,1} + \sigma_{12,2} = 0,\ \sigma_{12,1} + \sigma_{22,2} = 0; \tag{1.2}$$

$$\varepsilon_{11} = u_{1,1},\ \varepsilon_{22} = u_{2,2},\ \varepsilon_{12} = (1/2)(u_{1,2} + u_{2,1}); \tag{1.3}$$

$$\begin{gathered}
\sigma_{22} = q(x_1, t),\ \sigma_{12} = 0,\ x_2 = h, \\
u_2 = 0,\ \sigma_{12} = 0,\ x_2 = 0.
\end{gathered} \tag{1.4}$$

Here u_1 and u_2 are the displacements of points of the layer, and h is its thickness.

In order to seek an approximate solution, we shall expand the tangential stress σ_{12} into a Taylor series in x_2 in the neighborhood of the point $x_2 = 0$ and restrict ourselves to only linear terms [4], i.e., $\sigma_{12} = \varphi(x_1, t) + \psi(x_1, t)x_2$. Then from the conditions (1.4) $\sigma_{12} \equiv 0$, $\sigma_{12} \equiv 0$, and from the second Eq. (1.2) with (1.4) taken into account

*We shall consider a layer to be thin of the characteristic size of the region of its action loading is far greater than the layer thickness.

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$$\sigma_{22} = q(x_1, t). \tag{1.5}$$

Using the expressions for σ_{22} from (1.1) and (1.5), we find

$$\varepsilon_{22} = \frac{(1-2\nu)(1+\nu)}{1-\nu}(I-L)\frac{q(x_1,t)}{E} - \frac{\nu}{1-\nu}\varepsilon_{11}. \tag{1.6}$$

Now substituting (1.6) into (1.1), we obtain

$$\sigma_{11} = \frac{\nu}{1-\nu} q(x_1, t) + \frac{E}{1-\nu^2}(I+N)\varepsilon_{11}. \tag{1.7}$$

But from the first equilibrium equation $\sigma_{11} = \varphi_1(x_2, t)$ and

$$\varepsilon_{11} = -\nu(1+\nu)(I-L)q(x_1, t)/E \tag{1.8}$$

in accordance with (1.7).

Here $\varphi_1(x_2, t) \equiv 0$, since it is natural to assume that all the strains, stresses, and displacements are equal to zero when $q(x_1, t) \equiv 0$. In the following we shall immediately omit such functions without additional comments in similar situations.

Formulas (1.6) and (1.8) and the conditions (1.4) with (1.3) taken into account will give

$$\varepsilon_{22} = (1-\nu^2)(I-L)\frac{q(x_1,t)}{E},$$

$$u_2(x, t) = (1-\nu^2)\int_0^{x_2}(I-L)\frac{q(x_1,t)}{E}\,dx_2,$$

after which the stress-strain state of the layer is completely determined.

Problem 2. The action of a normal load $q(x_1, t)$ on a nonuniform viscoelastic thin layer bonded to a nondeformable base.

We shall write the boundary conditions of the problem in the form

$$\sigma_{22} = q(x_1, t),\ \sigma_{12} = 0,\ x_2 = h, \\ u_1 = 0,\ u_2 = 0,\ x_2 = 0. \tag{1.9}$$

Using the expansion of the tangential stress used previously, one can show that by virtue of the second condition (1.9) $\sigma_{12} = \psi'(x_1, t)/(x_2 - h)$ and then

$$\sigma_{22} = -\frac{(h-x_2)^2}{2}\psi''(x_1, t) + q(x_1, t)$$

follows from the second equilibrium equation (1.2) (we shall denote a derivative with respect to x_1 by a prime).

Thence and from the relationships (1.1) we determined

$$\varepsilon_{22} = \frac{(1-2\nu)(1+\nu)}{1-\nu}(I-L)\left[\frac{q(x_1,t)}{E} - \frac{(h-x_2)^2}{2E}\psi''(x_1,t)\right] - \frac{\nu}{1-\nu}\varepsilon_{11};$$

$$\sigma_{11} = \frac{\nu}{1-\nu}\left[q(x_1,t) - \frac{(h-x_2)^2}{2}\psi''(x_1,t)\right] + \frac{E}{1-\nu^2}(I+N)\varepsilon_{11}.$$

The first equilibrium equation (1.2) leads to the formula

$$\frac{E}{1-\nu^2}(I+N)\varepsilon_{11} + \psi(x_1,t) + \frac{\nu}{1-\nu}\left[q(x_1,t) - \frac{(h-x_2)^2}{2}\psi''(x_1,t)\right] = 0.$$

Taking account of the fact that for a thin layer $(h^2/2)\psi''(x_1, t) \ll \psi(x_1, t)$ and that the layer is rigidly bonded to a nondeformable base, i.e., $\varepsilon_{11} = 0$ at $x_2 = 0$, we obtain $\psi(x_1, t) = -\nu(1-\nu)^{-1}q(x_1, t)$. After this we arrive at the expressions

$$\sigma_{11} = \frac{\nu}{1-\nu} q(x_1, t), \quad \sigma_{12} = \frac{\nu}{1-\nu} q'(x_1, t)(h - x_2),$$
$$\sigma_{22} = q(x_1, t), \quad \varepsilon_{11} = 0, \quad \varepsilon_{22} = \frac{1-\nu-2\nu^2}{1-\nu}(I-L)\frac{q(x_1, t)}{E},$$
$$u_2(\mathbf{x}, t) = \frac{1-\nu-2\nu^2}{1-\nu}\int_0^{x_2}(I-L)\frac{q(x_1, t)}{E}\,dx_2$$

by neglecting terms of order h^2.

The values of u_1 and ε_{12} are easily determined from (1.3) and (1.1).

Problem 3. The action of a tangential load $\tau(x_1, t)$ on a nonuniform viscoelastic thin layer bonded to a nondeformable base.

For this problem we shall have the following boundary conditions:

$$\sigma_{22} = 0, \ \sigma_{12} = \tau(x_1, t), \ x_2 = h, \ u_1 = 0, \ u_2 = 0, \ x_2 = 0, \tag{1.10}$$

which when used along with the known expanison of the tangential stress permits obtaining $\sigma_{12} = \tau(x_1, t) + f'(x_1, t)(x_2 - h)$.

From the equilibrium equations (1.2)

$$\sigma_{11} = -f(x_1, t),$$
$$\sigma_{22} = (h - x_2)\tau'(x_1, t) - f''(x_1, t)\frac{(h-x_2)^2}{2}. \tag{1.11}$$

As before, from (1.1)

$$\varepsilon_{22} = \frac{(1-2\nu)(1+\nu)}{1-\nu}(I-L)\left[(h-x_2)\frac{\tau'(x_1, t)}{E} - \frac{(h-x_2)^2}{2E}f''(x_1, t)\right] - \frac{\nu}{1-\nu}\varepsilon_{11}; \tag{1.12}$$

$$\frac{E}{1-\nu^2}(I+N)\varepsilon_{11} - \frac{\nu}{1-\nu}\left[(x_2 - h)\tau'(x_1, t) + \frac{(h-x_2)^2}{2}f''(x_1, t)\right] + f(x_1, t) = 0. \tag{1.13}$$

Proceeding similarly to problem 2, we find from (1.13)

$$f(x_1, t) = -h\frac{\nu}{1-\nu}\tau'(x_1, t), \quad \varepsilon_{11} = \nu(1+\nu)x_2(I-L)\frac{\tau'(x_1, t)}{E}. \tag{1.14}$$

After this we write expressions for the remaining stresses and strains to within the accuracy of quantities containing h^2 in accordance with (1.11), (1.12). (1.14), (1.3), and (1.10):

$$\sigma_{11} = [\nu/(1-\nu)]h\tau'(x_1, t), \ \sigma_{12} = \tau(x_1, t), \ \sigma_{22} = (h - x_2)\tau'(x_1, t),$$
$$\varepsilon_{22} = [(1+\nu)/(1-\nu)](I-L)[(1-2\nu)h - (1-\nu)^2x_2][\tau'(x_1, t)/E],$$
$$u_1(\mathbf{x}, t) = \nu(1+\nu)x_2\int(I-L)[\tau'(x_1, t)/E]\,dx_1.$$

If the physiocomechanical characteristics of the medium do not depend on x_1, then

$$u_1(\mathbf{x}, t) = \nu(1+\nu)x_2(I-L)[\tau(x_1, t)/E].$$

2. Let us proceed to the solution of axisymmetric problems for a layer. We shall make use of a cylindrical coordinate system and the standard notation for it. In references to the formulas (1.1) we shall bear in mind that the appropriate rewriting of the notation is done, and in addition $x = x(r, z)$, and all the physicomechanical characteristics and the aging function depend only on z, i.e., $x = z$ in them. The equilibrium equations and the relations which relate the strains and stresses will take the form

$$\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0,\quad \frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0; \tag{2.1}$$

$$\varepsilon_r=\frac{\partial u}{\partial r},\ \varepsilon_\theta=\frac{u}{r},\ \varepsilon_z=\frac{\partial w}{\partial z},\ \varepsilon_{rz}=\frac{1}{2}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right]. \tag{2.2}$$

Problem 4. A normal load q(r, t) is acting on a thin nonuniform viscoelastic layer lying without friction in a rigid base.

The boundary conditions of the problem will be

$$\begin{gathered}\sigma_z=q(r,t),\ \tau_{rz}=0,\ z=h,\\ w=0,\ \tau_{rz}=0,\ z=0.\end{gathered} \tag{2.3}$$

In order to find an approximate solution, we shall expand the tangential stress into a Taylor series in the neighborhood of the point z = 0 and restrict ourselves to only linear terms, i.e.,

$$\tau_{rz}=\omega(r,t)+zp(r,t). \tag{2.4}$$

If we now take into account the boundary conditions (2.3), then we obtain τ_{rz} = 0 and σ_z = q(r, t). Thence with account taken of the expression for σ_z from (1.1), we find a relation among ε_z, ε_θ, and ε_r:

$$\varepsilon_z=\frac{(1-2\nu)(1+\nu)}{1-\nu}(I-L)\frac{q(r,t)}{E}-\frac{\nu}{1-\nu}(\varepsilon_\theta+\varepsilon_r). \tag{2.5}$$

With the help of (2.5) and (1.1) we obtain

$$\begin{gathered}\sigma_r=\frac{\nu}{1-\nu}q(r,t)+\frac{E}{1-\nu^2}(I+N)(\varepsilon_r+\nu\varepsilon_\theta),\\ \sigma_\theta=\frac{\nu}{1-\nu}q(r,t)+\frac{E}{1-\nu^2}(I+N)(\varepsilon_\theta+\nu\varepsilon_r).\end{gathered} \tag{2.6}$$

Then using (2.6) and the first equilibrium equation (2.1), we determine (we shall then denote derivatives with respect to r with a prime)

$$\frac{\varepsilon_r'+\nu\varepsilon_\theta'}{1-\nu}+\frac{\varepsilon_r-\varepsilon_\theta}{r}=-\frac{\nu(1+\nu)}{1-\nu}(I-L)\frac{q'(r,t)}{E}. \tag{2.7}$$

The relationship (2.7) with account taken of the fact that $(\varepsilon_r-\varepsilon_\theta)r^{-1}=\varepsilon_\theta'$ gives

$$\varepsilon_r+\varepsilon_\theta=-\nu(1+\nu)(I-L)[q(r,t)/E]. \tag{2.8}$$

Substituting the expressions (2.2) into (2.8), we obtain for our search for u:

$$\partial u/\partial r+u/r=-\nu(1+\nu)(I-L)[q(r,t)/E]. \tag{2.9}$$

Solving (2.9), we obtain

$$u=\frac{\Phi(z,t)}{r}-\nu(1+\nu)(I-L)r^{-1}\int\frac{q(r,t)}{E}rdr.$$

From notions of the boundedness of the displacement u at r = 0 the function $\Phi(z,t)\equiv 0$. Thence and from (2.2) and (2.6)

$$\begin{gathered}u=-\nu(1+\nu)(I-L)r^{-1}\int\frac{q(r,t)}{E}rdr,\\ \varepsilon_r=\nu(1+\nu)(I-L)\left[r^{-2}\int\frac{q(r,t)}{E}rdr-\frac{q(r,t)}{E}\right],\\ \varepsilon_\theta=-\nu(1+\nu)(I-L)r^{-2}\int\frac{q(r,t)}{E}rdr,\\ \sigma_r=\nu r^{-2}\int q(r,t)rdr,\quad \sigma_\theta=\nu\left[q-r^{-2}\int q(r,t)rdr\right].\end{gathered}$$

Formulas (2.5) and (2.8) and the conditions (2.3) permit determining

$$\varepsilon_z = (1-\nu^2)(I-L)\frac{q(r,t)}{E},$$

$$w(r,z,t) = (1-\nu^2)\int_0^z \frac{q(r,t)}{E(t+\varkappa(z),z)} - \int_{\tau_0}^t \frac{q(r,\tau)}{E(\tau+\varkappa(z),z)} K(t+\varkappa(z),$$
$$\tau+\varkappa(z),z)\,d\tau\,dz.$$

Problem 5. Now let a layer be rigidly bonded to a nondeformable base; then

$$\sigma_z = q(r,t),\ \tau_{rz} = 0,\ z = h, \qquad (2.10)$$
$$u = 0,\ w = 0,\ z = 0.$$

Using as before the representation (2.4) and the second condition (2.10), we obtain at z = h that $\tau_{rz} = (z-h)\varphi'(r,t)$, and from the second equilibrium equation

$$\sigma_z = -\frac{(h-z)^2}{2}\left[\varphi''(r,t) + \frac{\varphi'(r,t)}{r}\right] + q(r,t). \qquad (2.11)$$

Substituting σ_z from the relations (1.1) into (2.11), we find

$$\varepsilon_z = \frac{(1-2\nu)(1+\nu)}{1-\nu}(I-L)\frac{W(r,z,t)}{E} - \frac{\nu}{1-\nu}(\varepsilon_\theta+\varepsilon_r), \qquad (2.12)$$
$$W(r,z,t) = q(r,t) - \frac{(h-z)^2}{2}[\varphi''(r,t) + \varphi'(r,t)r^{-1}].$$

If we take the expression (2.12) into account in the relations (1.1) for σ_r and σ_θ, we obtain formulas similar to (2.6), in which one should replace q(r, t) by W(r, z, t). We shall substitute them into the first equilibrium equation; then we find

$$\varepsilon_r + \varepsilon_\theta = -(I-L)[\nu(1+\nu)W(r,z,t)/E + (1-\nu^2)\varphi(r,t)/E]. \qquad (2.13)$$

By virtue of the fact that we have a rigid seal, $\varepsilon_r = \varepsilon_\theta = 0$ at z = 0. In addition, taking into account that $(h^2/2)[\varphi''(r,t) + \varphi'(r,t)r^{-1}] \ll \varphi(r,t)$ for a thin layer and then neglecting quantities of the order h^2, we obtain from the relationship (2.13)

$$\varphi(r,t) = -\nu(1-\nu)^{-1}q(r,t),\ \varepsilon_r + \varepsilon_\theta \equiv 0. \qquad (2.14)$$

The expressions (2.14) and the notations of boundedness of u at r = 0 lead to the formulas

$$u = 0,\ \varepsilon_r = 0,\ \varepsilon_\theta = 0,\ \sigma_z = q(r,t),$$
$$\tau_{rz} = (h-z)\frac{\nu}{1-\nu}q'(r,t),\quad \sigma_r = \sigma_\theta = \frac{\nu}{1-\nu}q(r,t),$$
$$\varepsilon_z = \frac{(1-2\nu)(1+\nu)}{1-\nu}(I-L)\frac{q(r,t)}{E}.$$

Thence with account taken of the fact that w = 0 at z = 0, we obtain

$$w(r,z,t) = \frac{1-\nu-2\nu^2}{1-\nu}\int_0^z \frac{q(r,t)}{E(t+\varkappa(z),z)} - \int_{\tau_0}^t \frac{q(r,\tau)}{E(\tau+\varkappa(z),z)} K(t+\varkappa(z),$$
$$\tau+\varkappa(z),z)\,d\tau\,dz.$$

Problem 6. An axisymmetric tangential load $\tau(r, t)$ is acting on a layer; the layer is bonded to a nondeformable base.

Since the technical aspect of obtaining approximate solutions has been discussed in detail in the preceding problems, we shall give only the final results here:

$$
\begin{aligned}
&\sigma_r = \frac{\nu}{1-\nu} hY(r, t) - z\nu r^{-2}\int Y(r, t)\, r dr,\\
&\sigma_\theta = \frac{\nu}{1-\nu} hY(r, t) - \nu z\left[Y(r, t) - r^{-2}\int Y(r, t)\, r dr\right],\\
&\sigma_z = (h-z)\, Y(r, t), \quad \tau_{rz} = \tau(r, t),\\
&\varepsilon_r = \nu(1+\nu)(I-L)E^{-1}z\left[Y(r, t) - r^{-2}\int Y(r, t)\, r dr\right],\\
&\varepsilon_\theta = \nu(1+\nu)(I-L)E^{-1}zr^{-2}\int Y(r, t)\, r dr,\\
&\varepsilon_z = \frac{1+\nu}{1-\nu}(I-L)E^{-1}[(1-2\nu)h - (1-\nu)^2 z]\, Y(r, t),\\
&u = \nu(1+\nu)(I-L)E^{-1}zr^{-1}\int Y(r, t)\, r dr,\\
&Y(r, t) = \tau'(r, t) + r^{-1}\tau(r, t).
\end{aligned} \tag{2.15}
$$

The last entry in (2.15) is legitimate, since due to symmetry it is natural to assume that $\tau(r, t) \equiv 0$ at $r = 0$.

The results obtained show that in an elastic medium thin layers function under compression as a Fuss–Winkler base with pliability coefficients equal to $(1-\nu^2)hE^{-1}$ in problems 1 and 4 and to $(1-\nu-2\nu^2)(1-\nu)^{-1}hE^{-1}$ in problems 2 and 5. When functioning under shear a thin film can be treated as a Fuss–Winkler base with a pliability coefficient of $\nu(1+\nu)hE^{-1}$ only in the plane case (problem 3); this is not true now for the axisymmetric case (see (2.15)). We obtain some operator coefficients which operate on the applied load for the corresponding displacements of a selected model; in problems 3-5 this is only valid when the physico-mechanical properties of the layer vary only with depth.

The solutions given can be successfully used to calculate laminated bases with a complicated rheology if the characteristic size of the active loading zone of the upper layer is far larger than its thickness.

We note in conclusion that the solutions of nonlinear problems for a thin layer can be obtained in a similar way.

Plane problems for a thin layer under the conditions of established nonlinear creep were discussed in [4]. However, the algorithm proposed here permits obtaining more accurate solutions in the plane case (problem 3) and switching to the axisymmetric case.

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LITERATURE CITED

1. N. Kh. Arutyunyan, "Some problems of creep theory for nonuniformly aging bodies," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 3 (1976).
2. N. Kh. Arutyunyan, "Creep theory for nonuniformly hereditarily aging bodies," Dokl. Akad. Nauk SSSR, 229, No. 3 (1976).
3. N. Kh. Arutyunyan and B. A. Shoikhet, "Asymptotic behavior of the solution of the boundary-value problem of creep theory of nonuniformly aging bodies with unidirectional bonds," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 3 (1981).
4. M. A. Sumbatyan, "The plane problem for a thin layer under conditions of established nonlinear creep," Izv. Akad. Nauk SSSR, Mekhanika, 33, No. 1 (1980).